Lower bounds on the minimum distance of long codes in the Lee metric

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Outline

1. Motivation
2. Background on algebraic geometry codes
3. Gilbert type bound
4. Asymptotic rate of new constructible codes
5. Comparison
6. Conclusion

Motivation for Lee metric

Lee weight $wt_L(a)$ of a symbol $a \in \mathbb{Z}_q$, $wt_L(a) := \min(a, q-a)$,

Lee weight of $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_q^n$, $wt_L(\mathbf{x}) = \sum_{i=1}^{n} wt_L(x_i)$

- Application:
- • phase modulation (Berlekamp's book)
- • run length limited coding (Roth's book, Siegel's papers)
- Development of Theory
- • generalizing Hamming case (technical!)
- • giving contructible methods

Affine space vs projective space

- $n-$dimensional affine space over $\mathbb{F}_q$:

$$\mathbb{A}^n(\overline{\mathbb{F}_q}) := \{(x_1, x_2, \ldots, x_n) | x_i \in \overline{\mathbb{F}_q}\}.$$

- $n-$dimensional projective space over $\mathbb{F}_q$:

$$\mathbb{P}^n(\overline{\mathbb{F}_q}) := \left(\mathbb{A}^{n+1}(\overline{\mathbb{F}_q})\right)^* / \sim \; = \{[\mathbf{x}] = (x_1 : \cdots : x_{n+1}) | \mathbf{x} \in \mathbb{A}^{n+1}(\overline{\mathbb{F}_q})\}$$

with $\sim$ defined by:
$\forall \mathbf{a}, \mathbf{b} \in \mathbb{A}^{n+1}(\overline{\mathbb{F}_q}), \mathbf{a} \sim \mathbf{b}$, if $\exists \lambda \in \overline{\mathbb{F}_q}^*$, $\mathbf{a} = \lambda \mathbf{b}$.

Algebraic curves

Let F be an irreducible homogeneous polynomial in $\overline{\mathbb{F}_q}[X_1, X_2, \ldots, X_{n+1}]$

- A projective algebraic curve defined by F over $\mathbb{F}_q$ is

$$\mathcal{X} := \{(x_1 : \cdots : x_{n+1}) \in \mathbb{P}^n(\overline{\mathbb{F}_q}) | F(x_1, \ldots, x_{n+1}) = 0\},$$

- The zeros of F with coordinate x_i in $\mathbb{F}_q$ are called rational points.
- The zeros of F with the last coordinate 0 are called points at infinity.

Example

Let $F(X, Y, Z) = X^3 + XZ^2 + Z^3 + YZ^2 \in \overline{\mathbb{F}_2}[X, Y, Z]$.

- Then the plane projective curve defined by F is

$$\mathcal{X} = \{(x : y : z) \in \mathbb{P}^2(\overline{\mathbb{F}_2}) | F^*(x, y, z) = x^3 + xz^2 + z^3 + yz^2 = 0\}.$$

- There is only one rational point $(1 : 0 : 0)$
- There is only one point at infinity $(1 : 0 : 0)$.

Divisors

$\mathcal{X}$: an algebraic curve over $\mathbb{F}_q$

- Divisor on $\mathcal{X}$:

$$D := \sum_{P \in \mathcal{X}} n_P P$$

with $n_p \in \mathbb{Z}$ all zero except finite many

- Degree of D:

$$\deg(D) := \sum_{P \in \mathcal{X}} n_P \deg(P),$$

where $\deg(P) = |P^\sigma|$ with P^σ as orbit of P under $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

- $D = \sum_{P \in \mathcal{X}} n_P P \succcurlyeq D' = \sum_{P \in \mathcal{X}} n'_P P$ if $n_P \geq n'_P$ for all P.

Example

$\mathcal{X} = \{(x:y:z) \in \mathbb{P}^3(\overline{\mathbb{F}_2}) | x^3 + xz^2 + z^3 + yz^2 = 0\}$, a projective plane algebraic curve over $\mathbb{F}_2$

- points of degree 1 over $\mathbb{F}_2$:($x, y \in \mathbb{F}_2$)
  $P_\infty = (0:1:0)$
- points of degree 2 over $\mathbb{F}_2$:($x, y \in \mathbb{F}_{2^2} = \{0, 1, \omega, \bar{\omega}\}$)
  $P_1 = \{(0:\omega:1), (0:\bar{\omega}:1)\}$,
  $P_2 = \{(1:\omega:1), (1:\bar{\omega}:1)\}$,
  where $\omega, \bar{\omega}$ are roots of $y^2 + y = 1$ in $\mathbb{F}_{2^2}$.
- $D = 2P_1 + 3P_2 - 7P_\infty$: a divisor on $\mathcal{X}$
- $\deg(D) = 2.2 + 3.2 - 7.1 = 3$

Rational functions

Let $\mathcal{X}$ be an algebraic curve defined by F. A rational function on $\mathcal{X}$ is a function $f = g/h$ where f and g are homogeneous polynomials of the same degree with $g \notin \langle F \rangle$.

Rational divisors

Let f be a nonzero rational function on $\mathcal{X}$.

- A rational divisor of f: $\mathrm{div}(f) := \sum_{P \in \mathcal{X}} v_P(f)P$.
- $\mathrm{div}(f) = \sum_{P:\text{zero of } f} v_P(f)P - \sum_{P:\text{pole of } f} (-v_P(f))P$.
- $\deg(div(f)) = 0$.

Vector space associated with a divisor

Let G be a divisor on $\mathcal{X}$.

- Define $L(G) := \{f | f = 0 \text{ or } \mathrm{div}(f) + G \succcurlyeq \mathbf{0}\}$
- Dimension of $L(G)$ is denoted by $l(G)$.
- Genus of $\mathcal{X}$ is $\min\{g | l(G) \geq \deg(G) - g + 1\}$.

Consequence of Riemann-Roch Theorem

Let G be a divisor on an algebraic curve $\mathcal{X}$ having genus g. if $\deg(G) > 2g - 2$ then

$$l(G) = \deg(G) + 1 - g.$$

Definitions

- For two divisors G and $D = P_1 + P_2 + \cdots + P_n$ s.t $supp(D) \cap supp(G) = \emptyset$, $L(G) := \{f | f = 0 \textbf{ or } div(f) + G \geq 0\}$ $C(D, G) := \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in L(G)\}$, the algebraic geometry code
- $[n, k]_q$: linear (Lee) code of length n and dimension k over $\mathbb{F}_q$
- For a genus g, $N_q(g)$: the largest number of rational points
- $A(q) := \limsup_{g \to \infty} \frac{N_q(g)}{g}$, the Ihara function

Definitions

C_i: $[n_i, k_i]_q$ of Lee distance $d_L(C_i)$ such that $n_i \to \infty$.

- Rate: $R = \lim_{i\to\infty} \sup \frac{k_i}{n_i}$.
- Relative Lee distance: $\delta = \lim_{i\to\infty} \sup \frac{d_L(C_i)}{n_i s}$,

  with $s = \lfloor q/2 \rfloor$.

Asymptotic rates of AG codes

Theorem

There are families of geometric codes over $\mathbb{F}_{\mathcal{Q}}$ with rate $\mathcal{R}$ and relative Hamming distance Δ satisfying

$$\mathcal{R} + \Delta \geq 1 - \frac{1}{A(\mathcal{Q})}.$$

Asymptotic rate of AG code

- Theorem (Tsfasman-Vladut-Zink 1981)
  If $\mathcal{Q}$ is a square then
$$\mathcal{R} + \Delta \geq 1 - \frac{1}{\sqrt{\mathcal{Q}} - 1}.$$
- Theorem (Drinfeld-Vladut 1983)
  For any $\mathcal{Q}$,
$$A(\mathcal{Q}) \leq \sqrt{\mathcal{Q}} - 1.$$

Asymptotic rate of AG code

$$\mathcal{R} + \Delta \geq 1 - \frac{1}{A(\mathcal{Q})}.$$

- To get a lower bound for $\mathcal{R}$, we need the exact value of $A(\mathcal{Q})$ or a lower bound for $A(\mathcal{Q})$.
- If $\mathcal{Q}$ is a square then $A(\mathcal{Q}) = \sqrt{\mathcal{Q}} - 1$.
- For $\mathcal{Q}$ being prime, are there any methods to calculate $A(\mathcal{Q})$ or to lower-bound $A(\mathcal{Q})$?

Gilbert type bound

Theorem (Astola 1984)

If $q = 2s + 1$, *then* $R(\delta) \geq 1 + \log_q \alpha\beta^{\delta s}$, *where* α, β *are defined by*

$$\begin{aligned} \alpha + 2\alpha \sum_{i=1}^{s} \beta^i &= 1, \\ \alpha \sum_{i=1}^{s} i\beta^i &= \frac{\delta s}{2} \end{aligned}$$

Construction methods

- Concatenation
- Victoria
- Victoria+descent of the base field

Concatenation

Proposition

Let C_1 and C_2 be an $[N, K, D]_{q^k}$ and $[n, k]_q$ code with Lee distance d_L, respectively. Let Φ be a map defined by

$$\Phi : \mathbb{F}_{q^k} \longrightarrow C_2,$$

and

$$\Phi^* : (\mathbb{F}_{q^k})^N \longrightarrow C_2, \ s.t \ \Phi^*(v_1, \ldots, v_N) = (\Phi(v_1), \ldots, \Phi(v_N)).$$

Then $C = \Phi^*(C_1)$, called concatenated code, is an $[Nn, Kk]_q$ code with Lee distance Dd_L.

We call C_1 the outer code and C_2 the inner code.

Concatenation bound

- Proposition

  The rate R and the relative Lee distance δ of the concatenated code satisfy

$$\frac{R}{k/n} + \frac{\delta s}{d_L/n} \geq 1 - \frac{1}{q^{k/2} - 1}.$$

- Corollary

  For each prime $p \geq 7$ and every integer $1 \leq t \leq (p+1)/2$, such that p is congruent to $t+1 \mod 2$, there is a family of Lee codes over $\mathbb{Z}_p$ with rate R and relative Lee distance δ satisfying

$$\frac{R(p-1)}{p-1-t} + \frac{\delta s(p-1)}{2t} \geq 1 - \frac{1}{p^{(p-t-1)/2} - 1},$$

Victorian construction

- Take $G = rP$, i.e
  $C(D, rP) := \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in L(rP)\}$.
  Then
- • f has no pole except P whose order is at most r and
- • the number of zeros of f is at most r.
- The occurence of $f(P_i)$ in the codeword of $C(D, rP)$ is at most r times.
- Hence the minimum Lee distance d_L of $C(D, rP)$ is lower bounded by the Lee weight of a word whose entries are filled up with the first small Lee weights.

Victorian construction

Construct a word $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_q^n$ as follows:

- the first components a_i with $0^r, (\pm 1)^r, \ldots, (\pm M)^r$,
- the remaining components with $(M+1)$.
- Hence $d_L \geq wt_L(\mathbf{a})$.

Victorian bound

- Theorem (Wu-Kuijper-Udaya 2007)

  Given an algebraic curve of genus g over $\mathbb{F}_q$ having at least $n+1$ rational points, there are codes of parameters $[n-1, r-g]$ over $\mathbb{F}_q$ with Lee distance

  $$d_L \geq \frac{n^2 - r^2}{4r},$$

  for any integer r in the range $(2g-2, n)$.

- Corollary

  For a family of curves of genus $g \sim \gamma n$, the rate R of the attached family of codes of relative distance δ is

  $$R \geq -\gamma - 2\delta s + \sqrt{4\delta^2 s^2 + 1}.$$

Construction using descent of the base field

Let p be an odd prime and $\{1, \alpha\}$ a basis of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$.

- Then $\mathbb{F}_{p^2} = \mathbb{F}_p \cdot 1 + \mathbb{F}_p \cdot \alpha \cong \mathbb{F}_p \times \mathbb{F}_p$.
- We identify a word $c \in (\mathbb{F}_{p^2})^n$ with a word $\widetilde{c} \in (\mathbb{F}_p)^{2n}$.
- We identify an $[n, k]_{p^2}$ code C with an an $[2n, 2k]_p$ code $\widetilde{C}$.
- We extend the definition of the Lee weight to $\mathbb{F}_{p^2}$ by setting the weight of a symbol $z = x + y\alpha \in \mathbb{F}_{p^2}$ (where $x, y \in \mathbb{F}_p$) as

$$wt_L(z) = wt_L(x) + wt_L(y).$$

Construction using descent of the base field

The minimum Lee distance d_L of the $[n,k]_{p^2}$ code C is lower-bounded $wt_L(a)$ where $a = (a_1, a_2, \ldots, a_n) \in (\mathbb{F}_{p^2})^n$ is constructed as follows:

- all symbols $z \in \mathbb{F}_{p^2}$ of Lee weight $0, 1, \ldots, M$ occur in a exactly r times each
- some symbols of Lee weight $M+1$ could occur in a, but not more than r times each, and at least one of them less than r times
- no symbol of Lee weight greater than $M+1$ occur in a

Lower bounds on minimum Lee distance

Theorem

Let $M = \begin{cases} \left\lfloor \frac{1}{2}(-1+\sqrt{2n/r-1}) \right\rfloor & \text{if } 1 \leq n/r \leq \frac{p^2+4p-3}{2} \\ \left\lfloor p - \frac{1}{2}(1+\sqrt{2p^2+1-2n/r}) \right\rfloor & \text{if } \frac{p^2+4p-3}{2} < n/r \leq p^2. \end{cases}$

Then there are codes of parameters $[2(n-1), 2(r-g)]$ *over the prime field* $\mathbb{F}_p$ *with Lee distance* d_L *lower bounded by*

$$\begin{cases} (M+1)n + \frac{(M+1)(2M^2+4M+3)}{3} r & \text{if } n/r \leq \frac{p^2+4p-3}{2} \\ (M+1)n + \frac{2(M+1)(2M^2+4M-6pM-6p+3p^2)-p^3+p}{6} r & \text{if } n/r > \frac{p^2+4p-3}{2}. \end{cases}$$

Moreover, $d_L \geq \dfrac{n-r}{3}\sqrt{\dfrac{2n-r}{r}}$.

Lower bounds on rate

Corollary

Let $\gamma = \frac{1}{p-1}$, *R the code rate and δ relative distance. Then*

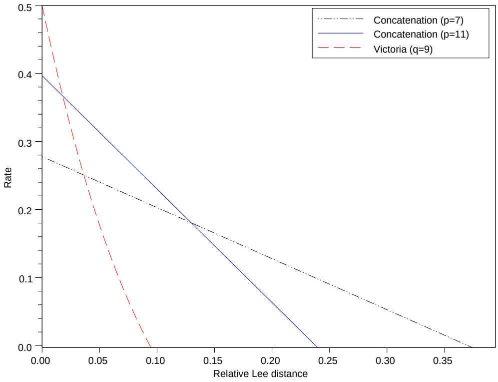
$$R \geq \begin{cases} 1 - 2\delta - \gamma & \text{if } 0 \leq \delta \leq \frac{2}{5} \quad (p \geq 3) \\ \frac{1}{3}(1-\delta) - \gamma & \text{if } \frac{2}{5} \leq \delta \leq \frac{10}{13} \quad (p \geq 5) \\ \text{etc.} & \\ c_M - d_M\delta - \gamma & \text{if } C(M) \leq \delta \leq C(M+1) \quad (p \geq 2M+3) \end{cases}$$

where $c_M = \frac{3}{2M^2+4M+3}$, $d_M = \frac{6}{(M+1)(2M^2+4M+3)}$, $C(M) = \frac{M+1}{2} - \frac{(M+1)(2M^2+4M+3)}{6(1+2M(M+1))}$.

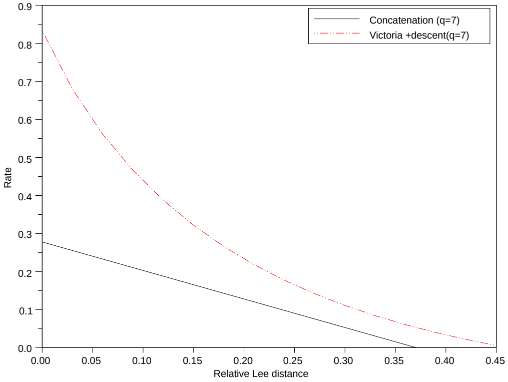
$$\text{Moreover}, R \geq \begin{cases} \left(\frac{-v-\sqrt{\Delta}}{2}\right)^{1/3} + \left(\frac{-v+\sqrt{\Delta}}{2}\right)^{1/3} + \frac{4}{3} - \gamma & \text{if } \Delta \geq 0 \\ 2\sqrt{\frac{-u}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(-\sqrt{\frac{27v^2}{-4u^3}}\right) + \frac{2\pi}{3}\right) + \frac{4}{3} - \gamma & \text{if } \Delta < 0 \end{cases}$$

with $\Delta = 6912\delta^6 + 2112\delta^4 - \frac{16\delta^2}{3}$, $u = 36\delta^2 - \frac{1}{3}$ *and* $v = (48\delta^2 - \frac{2}{7})$.

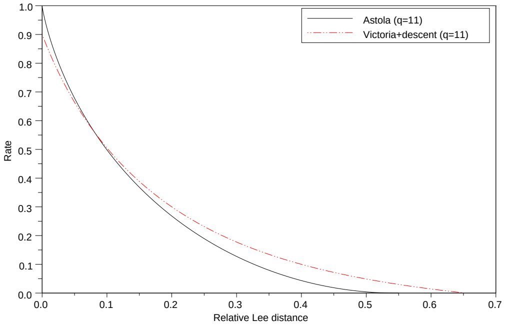
Concatenation vs Victoria



Concatenation vs Victoria+descent



Astola vs Victoria+descent



Thank you!